The twentieth workshop

What Comes Beyond the Standard Models?

July 9–17, 2017, Bled, Slovenia

K.V.Stepanyantz
Moscow State University, Physical Faculty, Department of Theoretical Physics

Structure of quantum corrections in $\mathcal{N}=1$ supersymmetric gauge theories

The Standard Model (SM) is a gauge theory with the group $SU(3)\times SU(2)\times U(1)$ and chiral fermions, which very successfully describes the strong and electroweak interactions. The gauge symmetry is spontaneously broken by the help of the Higgs mechanism,

$$SU(3)\times SU(2)\times U(1)\to SU(3)\times U(1)_{em},$$

producing the massive Z- and $W^{\pm}$-bosons together with the Higgs boson. Their experimental discoveries have been excellent confirmations of the Standard Model. From the theoretical point of view, it is a renormalizable model, in which anomalies introduced by chiral fermions cancel.

However, it is widely believed that the SM is not an ultimate theory.

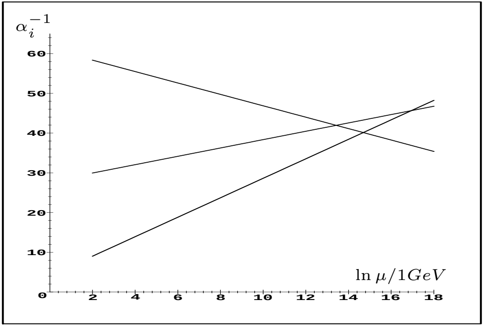
Quantum numbers of quarks and leptons imply that the gauge group $SU(3)\times SU(2)\times U(1)$ can be a remnant of a wider gauge group, e.g.,

$$SU(3)\times SU(2)\times U(1)\subset SU(5).$$

The $SO(10)$ group allows to place all fermions of one generation (including the right neutrinos) to the irreducible representation 16.

Incompleteness of the Standard Model

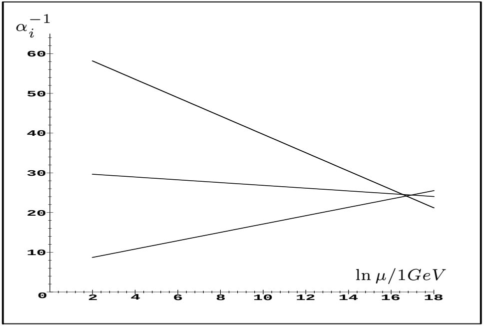
However, the evolution of running coupling constants in SM is not compatible with the prediction of Grand Unification Theories.
Moreover, the unification mass $M_X \sim 10^{15} GeV$ leads to unacceptably rapid proton decay.
From the theoretical point of view, quadratically divergent quantum corrections to the mass of the Higgs boson produce the problem of Higgs mass fine tuning.



A very promising way to solve the above problems is to consider $\mathcal{N}=1$ supersymmetric extensions of SM.

In the supersymmetric version of SM running of the gauge coupling constants agree with the predictions of Grand Unified Theories (GUT).

Increasing of the unification mass essentially increases the proton life time, $\tau \sim M_X^4$. There are no quadratically divergent quantum corrections to the Higgs mass.



Supersymmetric models predict a lot of new particles. There are superpartners of quarks, leptons, and gauge bosons. Supersymmetry also requires two Higgs doublets, which produces $2 \times 2 \times 2 - 3 = 5$ Higgs bosons.

To make masses of superpartners sufficiently large, it is necessary to break supersymmetry. Although it is highly desirable to break supersymmetry spontaneously, the simplest models (like MSSM) include soft terms, which explicitly break supersymmetry, but do not produce quadratic divergences. There are 4 types of the soft terms, including, e.g., gaugino masses.

In GUT theories running gaugino masses can be unified.

Investigation of quantum corrections in SUSY theories and theories with softly broken SUSY and comparing them with experimental data can give information about physics beyond SM.

It is convenient to describe $\mathcal{N}=1$ SUSY theories using $\mathcal{N}=1$ superfields, because in this case supersymmetry is manifest. In this language, the $\mathcal{N}=1$ SYM theory is given by the action

$$S=\frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mathrm{c.c.}\Big\},$$

where θ denotes an auxiliary Grassmannian coordinates. $V(x,\theta,\bar\theta)$ is the gauge superfield and the supersymmetric gauge field strength is defined as

$$W_a=\frac{1}{8}\bar D^2\left(e^{-2V}D_ae^{2V}\right).$$

$\phi_i(y^\mu,\theta)$ are chiral matter superfields, $\bar D_{\dot a}\phi_i=0$, where $y^\mu\equiv x^\mu+i\bar\theta^{\dot a}(\gamma^\mu)_{\dot a}{}^b\theta_b$ is the chiral coordinate and D_a and $\bar D_{\dot a}$ are the right and left supersymmetric covariant derivatives, respectively.

We assume that the theory is invariant under the gauge transformations

$$\phi \to e^{A}\phi; \qquad e^{2V} \to e^{-A^+}e^{2V}e^{-A},$$

where the parameter $A = ie_0 A^B T^B$ is an arbitrary chiral superfield. This gives the restrictions to the masses and Yukawa couplings,

$$m_0^{im}(T^A)_m{}^j + m_0^{mj}(T^A)_m{}^i = 0;$$

$$\lambda_0^{ijm}(T^A)_m{}^k + \lambda_0^{imk}(T^A)_m{}^j + \lambda_0^{mjk}(T^A)_m{}^i = 0.$$

For renormalizable theories the superpotential cannot be more than cubic in the chiral matter superfields.

It is well-known that ultraviolet behaviour of supersymmetric theories is better than in the non-supersymmetric case. For example, there are no quadratic divergences in $\mathcal{N}=1$ supersymmetric Yang–Mills theories (SYM) with matter. As we already mentioned, this is very impotent for phenomenology.

It is well known that the UV behavior of supersymmetric theories is better due to some non-renormalization theorems.

$\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory is finite in all orders.

Divergencies in $\mathcal{N} = 2$ SYM theories exist only in the one-loop approximation. $\mathcal{N} = 2$ hypermultiplets are not renormalized.

The superpotential in $\mathcal{N} = 1$ supersymmetric theories has no divergent quantum corrections.
The β-function of $\mathcal{N} = 1$ SYM theories is related to the anomalous dimension of the matter superfields by the so-called NSVZ β-function. For the pure $\mathcal{N} = 1$ SYM theory it gives the exact expression for the β-function in the form of the geometric series.
In this talk it will be also argued that in $\mathcal{N} = 1$ SYM theories the three-point ghost-gauge vertices are finite.

In $\mathcal{N} = 1$ supersymmetric theories the β-function is related to the anomalous dimension of the matter superfields by the equation

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}, \qquad \text{where}$$

$$\text{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2\delta^{AB}; \qquad r \equiv \delta_{AA}.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

The NSVZ β-function was obtained from different arguments: instantons, anomalies etc.

NSVZ β-function and calculations in the lowest loops

The NSVZ β-function can be compared with the results of calculations in the lowest orders of the perturbation theory. To make such calculations, a theory should be regularized.

The dimensional regularization breaks the supersymmetry and is not convenient for calculations in supersymmetric theories. That is why supersymmetric theories are mostly regularized by the dimensional reduction. However, the dimensional reduction is not self-consistent.

W.Siegel, Phys.Lett. **B 84** (1979) 193; **B 94** (1980) 37.

Removing of the inconsistencies leads to the loss of explicit supersymmetry:

L.V.Avdeev, G.A.Chochia, A.A.Vladimirov, Phys.Lett. **B 105** (1981) 272.

As a consequence, supersymmetry can be broken by quantum corrections in higher loops.

L.V.Avdeev, Phys.Lett. **B 117** (1982) 317;
L.V.Avdeev, A.A.Vladimirov, Nucl.Phys. **B 219** (1983) 262.

Using the dimensional reduction and $\overline{\text{DR}}$-scheme a β-function of $\mathcal{N}=1$ supersymmetric theories was calculated up to the four-loop approximation:

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **B 112** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett. **B 386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

The result coincides with the NSVZ β-function only in one- and two-loop approximations. In the higher loops it is necessary to make a special tuning of the coupling constant.

Thus, using of other regularizations is also interesting:

M.A.Shifman, A.I.Vainshtein, Sov.J.Nucl.Phys. **44** (1986) 321; J.Mas, M.Perez-Victoria, C.Seijas, JHEP, **0203** (2002) 049.

Usually in supersymmetric theories other regularizations are used for calculating a β-function only in one- and two-loop approximations.

The higher covariant derivative regularization is a consistent regularization, which does not break supersymmetry.

A.A.Slavnov, Nucl.Phys., **B 31** (1971) 301; Theor.Math.Phys. **13** (1972) 1064.

In order to regularize a theory by higher derivatives it is necessary to add a term with higher degrees of covariant derivatives. Then divergences remain only in the one-loop approximation. These remaining divergences are regularized by inserting the Pauli–Villars determinants.

A.A.Slavnov, Theor.Math.Phys. **33** (1977) 977.

The higher covariant derivative regularization can be generalized to the $\mathcal{N}=1$ supersymmetric case

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. **B 268** (1986) 113.

In this talk we will mostly discuss quantum corrections in SUSY theories regularized by higher covariant derivatives.

The simplest particular case of the $\mathcal{N}=1$ SYM theory is the $\mathcal{N}=1$ supersymmetric electrodynamics (SQED) with N_f flavors, which (in the massless case) is described by the action

$$S = \frac{1}{4e_0^2}\mbox{Re}\int d^4x\, d^2\theta\, W^a W_a + \sum_{f=1}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta\, \Big(\phi_f^* e^{2V}\phi_f + \widetilde\phi_f^* e^{-2V}\widetilde\phi_f\Big),$$

where V is a real gauge superfield, ϕ_f and $\widetilde\phi_f$ with $f=1,\ldots,N_f$ are chiral matter superfields, and $W_a = \bar D^2 D_a V/4$. This case corresponds to

$$C_2 = 0; \qquad C(R) = I; \qquad T(R) = 2N_f \qquad r = 1,$$

where I is the $2N_f\times 2N_f$ unit matrix. Therefore, for $\mathcal{N}=1$ SQED with N_f flavors the NSVZ β-function has the form

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. **B 166** (1986) 334.

In order to regularize the theory by higher derivatives, it is necessary to add the higher derivative term to the action:

$$S_{\text{reg}} = \frac{1}{4e_0^2}\text{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a + \sum_{f=1}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta \Big(\phi_f^* e^{2V}\phi_f + \widetilde{\phi}_f^* e^{-2V}\widetilde{\phi}_f\Big),$$

where $R(\partial^2/\Lambda^2)$ is a regulator, e.g. $R = 1 + \partial^{2n}/\Lambda^{2n}$.

Adding the higher derivative term allows to remove all divergences beyond the one-loop approximation. To remove one-loop divergences, we insert in the generating functional the Pauli–Villars determinants:

$$Z[J] = \int D\mu \prod_I \Big(\det PV(V, M_I)\Big)^{N_f c_I} \exp\Big\{ iS_{\text{reg}} + iS_{\text{gf}} + \text{Sources}\Big\},$$

$$\sum_I c_I = 1; \quad \sum_I c_I M_I^2 = 0; \quad M_I = a_I \Lambda, \quad \text{where } a_I \neq a_I(e_0).$$

$$\Gamma^{(2)} = \int \frac{d^4p}{(2\pi)^4}\, d^4\theta\, \Big(-\frac{1}{16\pi} V(-p)\,\partial^2\Pi_{1/2} V(p)\, d^{-1}(\alpha_0, \Lambda/p)$$
$$+\frac{1}{4}\sum_{f=1}^{N_f} \Big(\phi_f^*(-p,\theta)\phi_f(p,\theta) + \widetilde{\phi}_f^*(-p,\theta)\widetilde{\phi}_f(p,\theta)\Big)\, G(\alpha_0,\Lambda/p)\Big).$$

where $\partial^2\Pi_{1/2} \equiv -D^a \bar{D}^2 D_a/8$ is a supersymmetric transversal projection operator. The renormalized coupling constant $\alpha(\alpha_0, \Lambda/\mu)$ is defined by requiring that the inverse invariant charge $d^{-1}(\alpha_0(\alpha, \Lambda/\mu), \Lambda/p)$ is finite in the limit $\Lambda \to \infty$. The renormalization constant Z_3 is defined by

$$\frac{1}{\alpha_0} \equiv \frac{Z_3(\alpha, \Lambda/\mu)}{\alpha}.$$

The renormalization constant Z is constructed by requiring that the renormalized two-point Green function ZG is finite in the limit $\Lambda \to \infty$:

$$G_{\mathsf{ren}}(\alpha, \mu/p) = \lim_{\Lambda\to\infty} Z(\alpha, \Lambda/\mu) G(\alpha_0, \Lambda/p).$$

The renormalization group functions defined in terms of the bare coupling constant

In most original papers

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. **B 166** (1986) 334.

the NSVZ β-function was derived for the renormalization group functions defined in terms of the bare coupling constant

$$\beta\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv \frac{d\alpha_0(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}};$$

$$\gamma\Big(\alpha_0(\alpha,\Lambda/\mu)\Big) \equiv -\frac{d\ln Z(\alpha,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}}$$

These renormalization group functions
1. are scheme independent for a fixed regularization;
2. depend on the regularization;
2. in all loops satisfy the NSVZ relation in the case of $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives.

The renormalization group functions defined in terms of the bare coupling constant

The above RG functions do not depend on the renormalization prescription, because they can be expressed via unrenormalized Green functions:

$$0 = \lim_{p\to 0} \frac{d d^{-1}(\alpha_0, \Lambda/p)}{d\ln\Lambda}\Big|_{\alpha=\text{const}} = \lim_{p\to 0}\Big(\frac{\partial d^{-1}(\alpha_0,\Lambda/p)}{\partial\alpha_0}\beta(\alpha_0) - \frac{\partial d^{-1}(\alpha_0,\Lambda/p)}{\partial\ln p}\Big)$$

where in the last equality α_0 and p are considered as independent variables. Similarly, differentiating

$$\ln G(\alpha_0,\Lambda/q) = \ln G_{\text{ren}}(\alpha,\mu/q) - \ln Z(\alpha,\Lambda/\mu) + (\text{terms vanishing in the limit } q\to 0)$$

with respect to $\ln\Lambda$ at a fixed value of α, in the limit $q\to 0$ we obtain

$$\gamma(\alpha_0) = \lim_{q\to 0}\Big(\frac{\partial\ln G(\alpha_0,\Lambda/q)}{\partial\alpha_0}\beta(\alpha_0) - \frac{\partial\ln G(\alpha_0,\Lambda/q)}{\partial\ln q}\Big).$$

Therefore, $\beta(\alpha_0)$ and $\gamma(\alpha_0)$ do not depend on an arbitrariness of choosing the renormalization constants.

With the higher covariant derivative regularization loop integrals giving a β-function defined in terms of the bare coupling constant are integrals of total derivatives

A.Soloshenko, K.S., hep-th/0304083.

and even integrals of double total derivatives

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.

This allows to calculate one of the loop integrals analytically and to obtain the NSVZ relation for the RG functions defined in terms of the bare coupling constant. In the Abelian case this has been done in all loops

K.S., Nucl.Phys. **B 852** (2011) 71; JHEP **1408** (2014) 096.

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0}$$

$$= \frac{N_f}{\pi}\Big(1 - \frac{d}{d\ln\Lambda}\ln G(\alpha_0,\Lambda/q)\Big|_{q=0}\Big) = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = N_f \frac{d}{d\ln\Lambda}\Big\{2\pi\sum_I c_I \int \frac{d^4q}{(2\pi)^4}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\frac{\ln(q^2+M^2)}{q^2} + 4\pi\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{e^2}{k^2R_k^2}$$

$$\times\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big(\frac{1}{q^2(k+q)^2} - \sum_I c_I\frac{1}{(q^2+M_I^2)((k+q)^2+M_I^2)}\Big)\Big[R_k\Big(1+\frac{e^2N_f}{4\pi^2}\ln\frac{\Lambda}{\mu}\Big)$$

$$-2e^2N_f\Big(\int\frac{d^4t}{(2\pi)^4}\frac{1}{t^2(k+t)^2} - \sum_J c_J\int\frac{d^4t}{(2\pi)^4}\frac{1}{(t^2+M_J^2)((k+t)^2+M_J^2)}\Big)\Big]$$

$$+4\pi\int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{e^4}{k^2R_kl^2R_l}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big\{\Big(-\frac{2k^2}{q^2(q+k)^2(q+l)^2(q+k+l)^2}$$

$$+\frac{2}{q^2(q+k)^2(q+l)^2}\Big) - \sum_I c_I\Big(-\frac{2(k^2+M_I^2)}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)}$$

$$\times\frac{1}{((q+k+l)^2+M_I^2)} + \frac{2}{(q^2+M_I^2)((q+k)^2+M_I^2)((q+l)^2+M_I^2)} - \frac{1}{(q^2+M_I^2)^2}$$

$$\times\frac{4M_I^2}{((q+k)^2+M_I^2)((q+l)^2+M_I^2)}\Big)\Big\}$$

The integrals of double total derivatives do not vanish due to singularities of the integrand. This can be illustrated by a simple example

$$\int \frac{d^4q}{(2\pi)^4} \frac{\partial}{\partial q^\mu}\Big(\frac{q^\mu}{q^4} f(q^2)\Big) = -\frac{1}{8\pi^2} f(0),$$

where we assume that the function $f(q^2)$ has a sufficiently rapid fall-off at infinity. The sum of singular contributions appears to be proportional to the anomalous dimension.

Below we briefly explain, how one can prove the factorization of integrals defining the β-function into integrals of double total derivatives, following the method proposed in

K.S., Nucl.Phys. **B 852** (2011) 71.

It is convenient to use the background field method. In the Abelian case it is introduced by making the replacement

$$V \to V + \boldsymbol{V},$$

where $\boldsymbol{V}$ is the background gauge superfield.

1. The integral over the matter superfield is Gaussian and can be calculated exactly:

$$\exp\Big(i\Gamma[\boldsymbol{V}]\Big) = \int DV\, \prod_{I=0}^{m}\prod_{f=1}^{N_f} \det(\bigstar_{I,f})^{c_I/2} \exp\Big\{i\Big(S_{\text{gauge}} + S_\Lambda + S_{\text{gf}}\Big)\Big\},$$

where $c_0 = -1$, $M_0 = 0$ corresponds to usual superfields in the massless limit, and the operator $\bigstar$ encodes sequences of the vertices and propagators.

Terms quadratic in the superfield $\boldsymbol{V}$ in this expression have the form

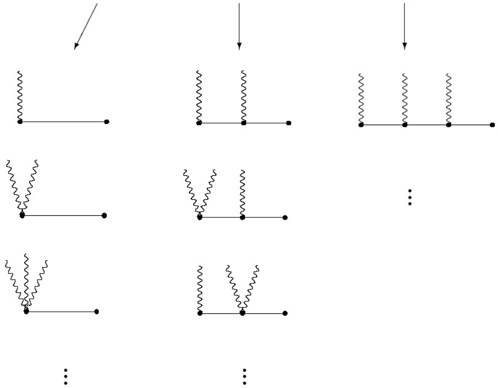
$$\Delta\Gamma^{(2)} = -\frac{i}{2}N_f^2\Big\langle\Big(\sum_{I=0}^{m} c_I\,\text{Tr}(\mathbf{V}QJ_0\star)_I\Big)^2\Big\rangle_{\text{1PI}}$$
$$+iN_f\sum_{I=0}^{m} c_I\Big\langle\text{Tr}(\mathbf{V}QJ_0\star\mathbf{V}QJ_0\star) + \text{Tr}(\mathbf{V}^2 J_0\star)\Big\rangle_{I,\text{1PI}},$$

where QJ_0 denotes the effective vertex and $\star$ does not contain $\boldsymbol{V}$.

Graphical interpretation of the operator $\star$

The operator $\star$ encodes sequences of vertices B and propagators P,

$$\star \;=\; 1 \;+\; BP \;+\; BPBP \;+\; BPBPBP \;+\; \ldots$$



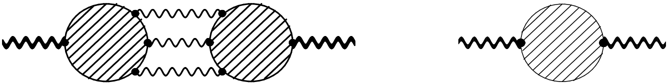
The trace makes a circle from the matter line.

Angular brackets denote the functional integration,

$$\langle A[V]\rangle \equiv \frac{\displaystyle\int DV\, A[V] \prod_{I=0}^{m}\prod_{f=1}^{N_f} \det(\star_I)^{c_I/2} \exp\Big\{i\Big(S_{\mathsf{gauge}} + S_\Lambda + S_{\mathsf{gf}}\Big)\Big\}}{\displaystyle\int DV \prod_{I=0}^{m}\prod_{f=1}^{N_f} \det(\star_I)^{c_I/2} \exp\Big\{i\Big(S_{\mathsf{gauge}} + S_\Lambda + S_{\mathsf{gf}}\Big)\Big\}}.$$

The angular brackets make propagators of the quantum gauge superfields V from V-s inside the operator $\star$.

Thus, non-trivial contributions to the effective action can appear from the following effective diagrams:



2. The β-function defined in terms of the bare coupling constant can be obtained from $\Delta\Gamma^{(2)}$ by the substitution

$$\boldsymbol{V}(x,\theta)\to\theta^4,$$

after which

$$\frac{1}{2\pi}\mathcal{V}_4\cdot\frac{\beta(\alpha_0)}{\alpha_0^2}=\frac{1}{2\pi}\mathcal{V}_4\cdot\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\Lambda/p)-\alpha_0\Big)\Big|_{p\to 0}=\frac{d(\Delta\Gamma^{(2)})}{d\ln\Lambda}\Big|_{\boldsymbol{V}=\theta^4},$$

where $\mathcal{V}_4$ is the (properly regularized) space-time volume.
After the above substitution we present the expression given earlier in the form of an integral of a double total derivative.
In the coordinate representation the integral of a total derivative is given by the expression

$$\mbox{Tr}\Big([x^\mu,\mbox{Something}]\Big)=0.$$

3. Diagrams in which the external lines are attached to different matter loops

After some algebraic transformations the corresponding contribution can be presented in the form

$$-\frac{i}{2}N_f^2\Big\langle\Big(\sum_{I=0}^{m} c_I\,\mbox{Tr}(\mathbf{V}QJ_0\star)_I\Big)^2\Big\rangle_{\mbox{1PI}}\Big|_{\boldsymbol{V}=\theta^4}$$
$$=\frac{i}{2}N_f^2\Big\langle\Big(\sum_{I=0}^{m} c_I\,\mbox{Tr}\,\bar\theta^{\dot a}(\gamma^\nu)_{\dot a}{}^b\theta_b\,\widetilde{Q}\,[y_\nu^*,\ln(\star_I)]\Big)^2\Big\rangle = 0,$$

where $y_\mu^* = x_\mu - i\bar\theta^{\dot a}(\gamma_\mu)_{\dot a}{}^b\theta_b$.

From this expression we see that the considered contribution is given by an integral of a double total derivative . It vanishes, because the integrand does not contain singularities.

4. The main (non-singlet) contribution

Similarly, the non-singlet contribution (which corresponds to diagrams in which external lines are attached to a single loop of the matter superfields) can be written as

$$i\frac{d}{d\ln\Lambda}N_f\sum_{I=0}^{m}c_I\Big\langle \mathrm{Tr}(\boldsymbol{V}QJ_0\star\boldsymbol{V}QJ_0\star)\Big\rangle_I\Big|_{\boldsymbol{V}=\theta^4}$$

$$= \text{One-loop result} - \frac{i}{2}N_f\,\frac{d}{d\ln\Lambda}\sum_{I=0}^{m}c_I\ \mathrm{Tr}\Big\langle\theta^4\Big[y_\mu^*,\Big[(y^\mu)^*,\ln(\star)\Big]\Big]\Big\rangle_I$$

$-$singular terms containing δ-functions,

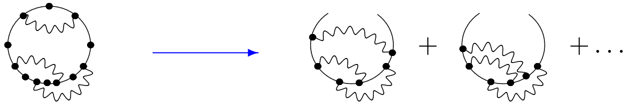
Thus, the sum of the considered diagrams is an integral of a double total derivative. However, the result does not vanish due to singularities of the integrand:

$$[x^\mu,\frac{\partial_\mu}{\partial^4}] = [-i\frac{\partial}{\partial P_\mu},-\frac{iP^\mu}{P^4}] = -2\pi^2\delta^4(P) = -2\pi^2 i\delta^4(p).$$

5. The sum of singularities

Evidently, singular contributions can be present only in the massless case. Therefore, there are no singularities for the Pauli–Villars superfields. Singularities proportional to δ-functions lead to cutting the diagrams (without external legs). As a result we obtain graphs corresponding to the two-point Green function of the matter superfields

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.



It is possible to relate the sum of singularities with the two-point Green function of the matter superfields.

6. The result

After summing singularities and adding the one-loop result we obtain

$$\frac{d\Delta\Gamma^{(2)}}{d\ln\Lambda}\Big|_{\boldsymbol{V}=\theta^4} = \frac{N_f}{2\pi^2}\mathcal{V}_4\cdot\Big(1-\frac{d\ln G}{d\ln\Lambda}\Big|_{q=0}\Big).$$

The expression

$$\frac{d\ln G}{d\ln\Lambda}\Big|_{q=0} = \frac{d}{d\ln\Lambda}\Big(\ln(ZG)-\ln Z\Big)\Big|_{q=0} = -\frac{d\ln Z}{d\ln\Lambda} = \gamma(\alpha_0)$$

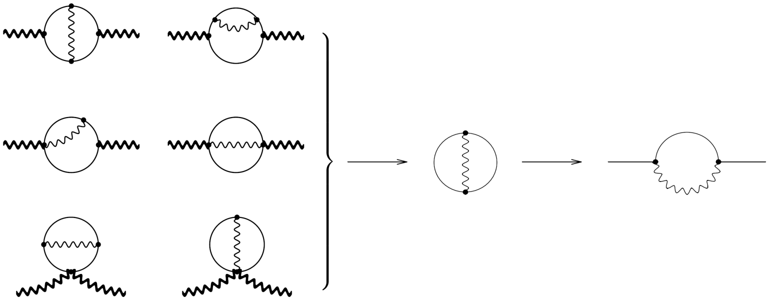
is the anomalous dimension (defined in terms of the bare coupling constant). Therefore, the final exact expression for the β-function defined in terms of the bare coupling constant for the considered theory has the form

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$

Derivation of the NSVZ β-function in the Abelian case by summing supergraphs

Qualitative picture:

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.



RG functions defined in terms of the bare coupling constant are scheme independent for a fixed regularization. However, RG functions are usually defined by a different way, in terms of the renormalized coupling constant,

$$\widetilde{\beta}\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\alpha(\alpha_0,\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}};$$

$$\widetilde{\gamma}\Big(\alpha(\alpha_0,\Lambda/\mu)\Big) \equiv \frac{d\ln Z(\alpha(\alpha_0,\Lambda/\mu),\Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}.$$

These RG functions are scheme-dependent. They coincide with the RG functions defined in terms of the bare coupling constant, if the boundary conditions

$$Z_3(\alpha,x_0)=1;\qquad Z(\alpha,x_0)=1$$

are imposed on the renormalization constants, where x_0 is an arbitrary fixed value of $\ln\Lambda/\mu$.

A.L.Kataev and K.S., Nucl.Phys. **B 875** (2013) 459; Phys.Lett. **B 730** (2014) 184; Theor.Math.Phys. **181** (2014) 1531.

$$\widetilde{\gamma}\left(\alpha(\alpha_0, x)\right) = -\frac{d\ln Z\left(\alpha(\alpha_0, x), x\right)}{dx}$$
$$= -\frac{\partial \ln Z(\alpha, x)}{\partial \alpha} \cdot \frac{\partial \alpha(\alpha_0, x)}{\partial x} - \frac{\partial \ln Z\left(\alpha(\alpha_0, x), x\right)}{\partial x},$$

where the total derivative with respect to $x = \ln \Lambda/\mu$ also acts on x inside α. Calculating these expressions at the point $x = x_0$ and taking into account that $\partial \ln Z(\alpha, x_0)/\partial\alpha = 0$ we obtain

$$\widetilde{\gamma}(\alpha_0) = \gamma(\alpha_0).$$

The equality for the β-functions can be proved similarly.
The RG functions $\widetilde{\beta}$ and $\widetilde{\gamma}$ (defined in terms of the renormalized coupling constant) are scheme-dependent. They satisfy the NSVZ relation only in a certain subtraction scheme, called the NSVZ scheme, which is evidently fixed in all loops by the boundary conditions

$$(Z_3)_{\mbox{NSVZ}}(\alpha_{\mbox{NSVZ}}, x_0) = 1; \qquad Z_{\mbox{NSVZ}}(\alpha_{\mbox{NSVZ}}, x_0) = 1,$$

if the theory is regularized by higher derivatives.

The (three-loop) renormalized coupling constant for $\mathcal{N}=1$ SQED can be calculated in the case $R_k = 1 + k^{2n}/\Lambda^{2n}$:

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big) - \frac{\alpha^2 N_f}{\pi^3}\Big(\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu} - \ln\frac{\Lambda}{\mu}\Big(N_f\sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2} - N_f b_1\Big) + b_3\Big) + O(\alpha^3),$$

where b_i are arbitrary finite constants.

Similarly, the renormalization constant Z (in the two-loop approximation) for the matter superfields is not also uniquely defined:

$$Z = 1 + \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{\alpha^2(N_f+1)}{2\pi^2}\ln^2\frac{\Lambda}{\mu} - \frac{\alpha^2}{\pi^2}\ln\frac{\Lambda}{\mu}\Big(N_f\sum_{I=1}^{n} c_I\ln a_I - N_f b_1 + N_f + \frac{1}{2} - g_1\Big) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3),$$

where g_i are other arbitrary finite constants.

The subtraction scheme is fixed by values of the constants b_i and g_i.

The RG functions defined in terms of the bare coupling constant are

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3}\Big(N_f\sum_{I=1}^n c_I \ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha_0^3);$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2}\Big(N_f\sum_{I=1}^n c_I \ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha_0^3).$$

They do not depend on the finite constants b_i and g_i (i.e. they are scheme-independent) and satisfy the NSVZ relation.

The RG functions defined in terms of the renormalized coupling constant are

$$\frac{\widetilde\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f\sum_{I=1}^n c_I \ln a_I + N_f + \frac{1}{2} + N_f(b_2 - b_1)\Big) + O(\alpha^3)$$

$$\widetilde\gamma(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{1}{2} + N_f\sum_{I=1}^n c_I \ln a_I - N_f b_1 + N_f g_1\Big) + O(\alpha^3)$$

and depend on a subtraction scheme.

The NSVZ scheme is determined by the conditions

$$\alpha_0(\alpha_{\text{NSVZ}}, x_0) = \alpha_{\text{NSVZ}}; \qquad Z_{\text{NSVZ}}(\alpha_{\text{NSVZ}}, x_0) = 1$$

For simplicity we set $g_1 = 0$ (this constant can be excluded by a redefinition of μ). In this case $x_0 = 0$ and the above conditions (for the NSVZ scheme) give

$$g_2 = b_1 = b_2 = b_3 = 0.$$

In this case in the considered approximations

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f \sum_{I=1}^{n} c_I \ln a_I + N_f + \frac{1}{2}\Big) + O(\alpha^3) = \frac{\beta(\alpha)}{\alpha^2};$$

$$\widetilde{\gamma}(\alpha) = \frac{d\ln Z}{d\ln\mu} = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{1}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I\Big) + O(\alpha^3) = \gamma(\alpha).$$

Consequently, in this scheme the NSVZ relation is satisfied.

NSVZ-scheme with the higher derivatives

$$\widetilde\gamma_{\mathrm{NSVZ}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I + N_f\Big) + O(\alpha^3);$$

$$\widetilde\beta_{\mathrm{NSVZ}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \sum_{I=1}^{n} c_I \ln a_I + N_f\Big) + O(\alpha^3)\Big).$$

MOM-scheme (The results with the dimensional reduction and with the higher derivative regularization coincide.)

$$\widetilde\gamma_{\mathrm{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde\beta_{\mathrm{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1-\zeta(3))\Big) + O(\alpha^3)\Big).$$

$\overline{\mathrm{DR}}$-scheme

$$\widetilde\gamma_{\overline{\mathrm{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde\beta_{\overline{\mathrm{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

In the $\overline{\text{DR}}$-scheme the NSVZ relation is not valid starting from the three-loop approximation

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **B 112** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett. **B 386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

due to the scheme-dependence.

Why the higher derivative regularization naturally gives NSVZ and the dimensional reduction does not?

In the above derivation we essentially use the possibility to take the limit $p \to 0$. This follows from the fact that the higher derivative terms and the derivative with respect to $\ln\Lambda$ make the integrals in this limit well-defined. In the case of using the dimensional reduction the limit $p \to 0$ is not well-defined. However, it is possible to make calculations similar to the case of using the higher derivative regularization

S.S.Aleshin, A.L.Kataev, K.S., JETP Lett. **103** (2016) 77; S.S.Aleshin, I.O.Goriachuk, A.L.Kataev, K.S., Phys.Lett. B764 (2017) 222.

With the dimensional reduction in the three loop approximation

$$d^{-1}(\alpha_0,\Lambda/p,\varepsilon)-\alpha_0^{-1}=8\pi N_f\Lambda^{\varepsilon}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+p)^2}$$

$$-8\pi N_f\Lambda^{\varepsilon}\frac{\varepsilon}{1-\varepsilon}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+p)^2}\big(\ln G(\alpha_0,q/\Lambda,\varepsilon)\big)_{\text{1-loop}}$$

$$-8\pi N_f\Lambda^{\varepsilon}\frac{2\varepsilon}{1-3\varepsilon/2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+p)^2}\big(\ln G(\alpha_0,q/\Lambda,\varepsilon)\big)_{\text{2-loop},N_f}$$

$$+\text{finite terms}+O(\alpha_0^2N_f)+O(\alpha_0^3),$$

Then the boundary conditions analogous to the case of HD (at the three-loop level) are

$$\lim_{\varepsilon\to\infty}\alpha_0(\alpha',\varepsilon,x_0=0)=\alpha'-\frac{\alpha'^3N_f}{4\pi^2}+O(\alpha'^4);\qquad \lim_{\varepsilon\to\infty}Z'(\alpha',\varepsilon,x_0=0)=1.$$

They are equivalent to the coupling constant redefinition ($\alpha'\sim$ NSVZ and $\alpha\sim\overline{\text{DR}}$)

$$\alpha'=\alpha+\frac{\alpha^3N_f}{4\pi^2}+O(\alpha^4).$$

The integrals defining the anomalous dimension of the photino mass

$$\gamma_m(\alpha_0) \equiv \frac{d\ln m_0}{d\ln\Lambda}$$

in softly broken $\mathcal{N}=1$ SQED regularized by higher derivatives are also integrals of double total derivatives in all loops.

I.V.Nartsev, K.S., JHEP **1704** (2017) 047; JETP Lett. **105** (2017) 69.

This can be proved by the generalization of the method described above and leads to the NSVZ-like relation

$$\gamma_m(\alpha_0) = \frac{\alpha_0 N_f}{\pi}\Big[1 - \frac{d}{d\alpha_0}\Big(\alpha_0\gamma(\alpha_0)\Big)\Big].$$

J.Hisano, M.A.Shifman, Phys.Rev. **D56** (1997) 5475;
I.Jack, D.R.T.Jones, Phys.Lett. **B415** (1997) 383;
L.V.Avdeev, D.I.Kazakov, I.N.Kondrashuk, Nucl.Phys. **B510** (1998) 289.

The NSVZ-like scheme (for the RG functions defined in terms of the renormalized coupling constant) in this case is defined by the conditions

$$Z_3(\alpha, x_0) = 1; \qquad Z(\alpha, x_0) = 1; \qquad Z_m(\alpha, x_0) = 1.$$

Simple non-Abelian example: exact expression for the Adler D-function in $\mathcal{N}=1$ SQCD

M.A.Shifman and K.S., Phys.Rev.Lett. **114** (2015) 051601; Phys.Rev. **D 91** (2015) 105008.

Let us consider $\mathcal{N}=1$ SQCD interacting with the Abelian gauge field. This theory is described by the action

$$S=\frac{1}{2g_0^2}\text{tr}\,\text{Re}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4e_0^2}\text{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{f=1}^{N_f}\Big[\frac{1}{4}\int d^4x\,d^4\theta\Big(\phi_f^+e^{2q_f\boldsymbol{V}+2V}\phi_f+\widetilde\phi_f^+e^{-2q_f\boldsymbol{V}-2V^t}\widetilde\phi_f\Big)$$
$$+\Big(\frac{1}{2}\int d^4x\,d^2\theta\,m_{0f}\widetilde\phi_f^t\phi_f+\text{c.c.}\Big)\Big].$$

We assume that the gauge group is $SU(N_c)$, and matter superfields belong to the (anti)fundamental representation.
This theory is a supersymmetric generalization of QCD, in which one takes into account interaction of quarks with the electromagnetic field.

V is the non-Abelian $SU(N_c)$ gauge superfield (gluons + superpartners)
$\boldsymbol{V}$ is the Abelian $U(1)$ gauge superfield (photon + superpartner)
ϕ_f and $\widetilde{\phi}_f$ are the chiral matter superfields with the charges $q_f e$ and $-q_f e$ with respect to the group $U(1)$, respectively (right and left quarks + superpartners).
The strength of the non-Abelian gauge superfield is denoted by

$$W_a \equiv \frac{1}{8}\bar{D}^2(e^{-2V}D_a e^{2V}),$$

and the strendgth of the Abelian gauge superfield is

$$\boldsymbol{W}_a = \frac{1}{4}\bar{D}^2 D_a \boldsymbol{V}.$$

The considered theory contains two coupling constants:

$$\alpha_s = \frac{g^2}{4\pi} \qquad \text{and} \qquad \alpha = \frac{e^2}{4\pi}.$$

We consider quantum corrections to the electromagnetic coupling contant α, which appear due to the quark loop with internal gluon and quark lines. The diagrams containing internal photon lines are omitted. (Thus, the electromagnetic field $\boldsymbol{V}$ is considered as an external field.)
Due to the Ward identity the two-point Green function of the superfield $\boldsymbol{V}$ is transversal:

$$\Delta\Gamma^{(2)} = -\frac{1}{16\pi}\int \frac{d^4p}{(2\pi)^4}\, d^4\theta\, \boldsymbol{V}\, \partial^2\Pi_{1/2}\boldsymbol{V}\,\Big(d^{-1}(\alpha_0,\alpha_{0s},\Lambda/p) - \alpha_0^{-1}\Big).$$

We calculate the Adler function, which is defined in terms of the bare coupling constant by the equation

$$D(\alpha_{0s}) = \frac{3\pi}{2}\frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\alpha_{0s},\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0} = \frac{3\pi}{2\alpha_0^2}\frac{d\alpha_0}{d\ln\Lambda}.$$

Thus, it depends on regularization, but is independent of a subtraction scheme.

We add to the action the higher derivative term, e.g.,

$$S_\Lambda = \frac{1}{2g_0^2}\mbox{tr}\,\mbox{Re}\int d^4x\,d^2\theta\,(e^\Omega W^a e^{-\Omega})\Big[R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)-1\Big](e^\Omega W_a e^{-\Omega}).$$

The covariant derivatives have the form

$$\nabla_a = e^{-\Omega^+}D_a e^{\Omega^+};\qquad \bar\nabla_{\dot a} = e^{\Omega}\bar D_{\dot a}e^{-\Omega},\qquad \mbox{where}\qquad e^{2V} = e^{\Omega^+}e^{\Omega},$$

Λ is a dimensionful parameter, and $R-1$ is a regulator, such as $R(0)-1=0$ and $R(x)\to\infty$ for $x\to\infty$, for example, $R(x)=1+x^n$.
Remaining one-loop (sub)divergences are regularized by inserting the Pauli–Villars determinants into the generating functional:

$$\Gamma[\boldsymbol{V}] = -i\ln\int DV\,D\Phi\,D\widetilde\Phi\,\prod_{I=1}^m \det(V,\boldsymbol{V},M_I)^{c_I}\exp\Big(i(S+S_\Lambda+S_{\mbox{gf}}+S_{\mbox{ghosts}})\Big),$$

where $M_I = a_I\Lambda$ and a_I do not depend on α_{0s}.

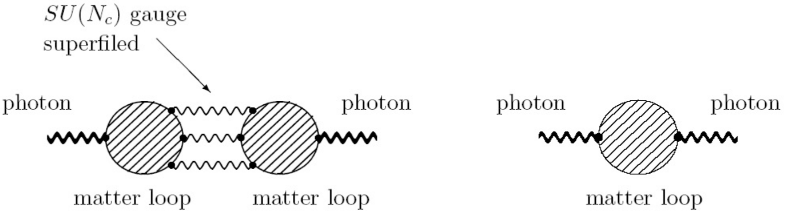
It is possible to derive the following NSVZ-like exact expression for the Adler function for the considered theory

$$D(\alpha_{0s}) = \frac{3}{2}\sum_f q_f^2 \cdot N_c\Big(1-\gamma(\alpha_{0s})\Big).$$

Note that, in general, the Adler D-function consists of two contributions

$$D(\alpha_{0s}) = \sum_f q_f^2\, D_1(\alpha_{0s}) + \Big(\sum_f q_f\Big)^2 D_2(\alpha_{0s}),$$

which correspond to two different types of diagrams:



Investigation of non-Abelian $\mathcal{N}=1$ SYM theories with matter is much more complicated. Let us consider the theory

$$S=\frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mbox{c.c.}\Big\},$$

where matter superfields belong to a representation R of the gauge group, and Yukawa couplings λ_0 satisfy the condition

$$\lambda_0^{ijm}(T^A)_m{}^k+\lambda_0^{imk}(T^A)_m{}^j+\lambda_0^{mjk}(T^A)_m{}^i=0.$$

It is invariant under the gauge transformations

$$\phi\to e^A\phi;\qquad e^{2V}\to e^{-A^+}e^{2V}e^{-A},$$

where the parameter A is an arbitrary chiral superfield.

Quantum-background splitting is made by the substitution

$$e^{2V} \to e^{\boldsymbol{\Omega}^+} e^{2V} e^{\boldsymbol{\Omega}}.$$

The background superfield $\boldsymbol{V}$ is defined by $e^{2\boldsymbol{V}} = e^{\boldsymbol{\Omega}^+} e^{\boldsymbol{\Omega}}$.
We choose the following higher derivative term

$$S_\Lambda = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, e^{\Omega} e^{\boldsymbol{\Omega}} W^a e^{-\boldsymbol{\Omega}} e^{-\Omega} \Big[R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)-1\Big]_{Adj}$$
$$\times e^{\Omega} e^{\boldsymbol{\Omega}} W_a e^{-\boldsymbol{\Omega}} e^{-\Omega} + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^+ e^{\boldsymbol{\Omega}^+} e^{\Omega^+}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)-1\Big] e^{\Omega} e^{\boldsymbol{\Omega}}\phi,$$

and the gauge fixing term

$$S_{\mbox{gf}} = \frac{1}{e_0^2}\mbox{tr}\int d^4x\, d^4\theta\, \Big(16\xi_0\, f^+ \Big[e^{\boldsymbol{\Omega}^+} K^{-1}\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big) e^{\boldsymbol{\Omega}}\Big]_{Adj} f$$
$$+ e^{\boldsymbol{\Omega}} f e^{-\boldsymbol{\Omega}} \boldsymbol{\nabla}^2 V + e^{-\boldsymbol{\Omega}^+} f^+ e^{\boldsymbol{\Omega}^+} \bar{\boldsymbol{\nabla}}^2 V\Big),$$

where the regulators R, F, and K have a rapid growth at infinity.

Actions for the Faddeev–Popov and Nielsen–Kallosh ghosts have the form

$$S_{\text{FP}} = \frac{1}{e_0^2}\text{tr}\int d^4x\, d^4\theta\, \Big(e^{\boldsymbol{\Omega}}\bar{c}e^{-\boldsymbol{\Omega}} + e^{-\boldsymbol{\Omega}^+}\bar{c}^+e^{\boldsymbol{\Omega}^+}\Big)$$
$$\times\Big\{\Big(\frac{V}{1-e^{2V}}\Big)_{Adj}\Big(e^{-\boldsymbol{\Omega}^+}c^+e^{\boldsymbol{\Omega}^+}\Big) + \Big(\frac{V}{1-e^{-2V}}\Big)_{Adj}\Big(e^{\boldsymbol{\Omega}}ce^{-\boldsymbol{\Omega}}\Big)\Big\};$$
$$S_{\text{NK}} = \frac{1}{2e_0^2}\text{tr}\int d^4x\, d^4\theta\, b^+\Big[e^{\boldsymbol{\Omega}^+}K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)e^{\boldsymbol{\Omega}}\Big]_{Adj} b.$$

The total action of the gauge fixed theory is invariant under the BRST transformations

$$\delta V = -\varepsilon\Big\{\Big(\frac{V}{1-e^{2V}}\Big)_{Adj}\Big(e^{-\boldsymbol{\Omega}^+}c^+e^{\boldsymbol{\Omega}^+}\Big) + \Big(\frac{V}{1-e^{-2V}}\Big)_{Adj}\Big(e^{\boldsymbol{\Omega}}ce^{-\boldsymbol{\Omega}}\Big)\Big\};$$

$$\delta\phi = \varepsilon c\phi; \qquad \delta\bar{c} = \varepsilon\bar{D}^2(e^{-2\boldsymbol{V}}f^+e^{2\boldsymbol{V}}); \qquad \delta\bar{c}^+ = \varepsilon D^2(e^{2\boldsymbol{V}}fe^{-2\boldsymbol{V}});$$
$$\delta c = \varepsilon c^2; \qquad \delta c^+ = \varepsilon(c^+)^2; \qquad \delta f = 0; \qquad \delta b = 0; \qquad \delta\boldsymbol{\Omega} = 0,$$

where ε is an anticommuting scalar parameter.

In our notation the renormalization constants are defined by the equations

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \frac{1}{\xi_0} = \frac{Z_\xi}{\xi}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R;$$

$$b = \sqrt{Z_b}\, b_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1} \bar{c}_R c_R; \qquad \phi_i = (\sqrt{Z_\phi})_i{}^j (\phi_R)_j;$$

$$m^{ij} = m_0^{mn} (Z_m)_m{}^i (Z_m)_n{}^j; \qquad \lambda^{ijk} = \lambda_0^{mnp} (Z_\lambda)_m{}^i (Z_\lambda)_n{}^j (Z_\lambda)_p{}^k.$$

The subscript R denotes renormalized superfields, α, λ, and ξ are the renormalized coupling constant, the Yukawa couplings, and the gauge parameter, respectively; m denotes renormalized masses.
It is possible to impose the following constrains to these renormalization constants:

$$(Z_m)_i{}^j = (Z_\lambda)_i{}^j = (\sqrt{Z_\phi})_i{}^j; \qquad Z_\xi = Z_V^{-2}; \qquad Z_b = Z_\alpha^{-1}.$$

Non-renormalization of the vertices with two ghost legs and one leg of the quantum gauge superfield

We will prove that the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield are finite in all orders.

K.S., Nucl.Phys. **B909** (2016) 316.

There are 4 such vertices, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$.
They have the same renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, the above statement can be rewritten as

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0.$$

In the one-loop approximation this has first been noted in the paper

S.S.Aleshin, A.E.Kazantsev, M.B.Skopsov, K.S., JHEP **1605** (2016) 014.

Consequently, there is a subtraction scheme in which

$$-\frac{1}{2}\ln Z_\alpha + \ln Z_c + \ln Z_V = 0.$$

Important: Below we will demonstrate that Z_c is divergent. Therefore, The Green functions of the structure $\bar{c}V^nc$ are divergent for $n \neq 1$.

The Slavnov–Taylor identity can be derived by making the substitution coinciding with the BRST transformations in the generating functional and is written as

$$0 = \int d^4x\, d^4\theta_x\, \frac{\delta\Gamma}{\delta V_x^A} \left\langle \delta V_x^A \right\rangle + \int d^4x\, d^2\theta_x \Big(\left\langle \delta\bar{c}_x^A \right\rangle \frac{\delta\Gamma}{\delta\bar{c}_x^A} + \left\langle \delta c_x^A \right\rangle \frac{\delta\Gamma}{\delta c_x^A}$$

$$+ \left\langle \delta\phi_i \right\rangle \frac{\delta\Gamma}{\delta\phi_i} \Big) + \int d^4x\, d^2\bar{\theta}_x \Big(\left\langle \delta\bar{c}_x^{*A} \right\rangle \frac{\delta\Gamma}{\delta\bar{c}_x^{*A}} + \left\langle \delta c_x^{*A} \right\rangle \frac{\delta\Gamma}{\delta c_x^{*A}} + \left\langle \delta\phi^{*i} \right\rangle \frac{\delta\Gamma}{\delta\phi^{*i}} \Big),$$

where we keep the ε-dependence.

Also we will use the identity obtained by making the substitution $\bar{c} \to \bar{c} + a$, where a is an arbitrary chiral superfield:

$$\varepsilon \frac{\delta\Gamma}{\delta\bar{c}_x^A} = \frac{1}{4}\bar{D}^2 \left\langle \delta V_x^A \right\rangle; \qquad \varepsilon \frac{\delta\Gamma}{\delta\bar{c}_x^{*A}} = \frac{1}{4} D^2 \left\langle \delta V_x^A \right\rangle,$$

where, for simplicity, the background superfield is set to 0.

Let us differentiate the Slavnov–Taylor identity with respect to $\bar{c}^{*B}_y$, c^C_z, and c^D_w, set the fields to 0, and use the equations

$$\frac{\delta^2\Gamma}{\delta\bar{c}^{*B}_y\delta c^A_x} = -\frac{D^2_y\bar{D}^2_x}{16}G_c\delta^8_{xy}\delta_{AB}; \qquad \frac{\delta}{\delta c^A_x}\left\langle\delta V^B_y\right\rangle = -\varepsilon\cdot\frac{1}{4}G_c\,\bar{D}^2\delta^8_{xy}\delta_{AB}.$$

As a result we obtain the identity

$$\varepsilon\cdot G_c(\partial^2_w/\Lambda^2)\bar{D}^2_w\frac{\delta^3\Gamma}{\delta\bar{c}^{*B}_y\delta V^D_w\delta c^C_z} - \varepsilon\cdot G_c(\partial^2_z/\Lambda^2)\bar{D}^2_z\frac{\delta^3\Gamma}{\delta\bar{c}^{*B}_y\delta V^C_z\delta c^D_w}$$
$$+\frac{1}{2}G_c\left(\partial^2_y/\Lambda^2\right)D^2_y\frac{\delta^2}{\delta c^C_z\delta c^D_w}\left\langle\delta c^B_y\right\rangle = 0.$$

Similarly, differentiating with respect to $\bar{c}^{*B}_y$, c^{*C}_z, and c^D_w gives

$$\varepsilon\cdot G_c(\partial^2_w/\Lambda^2)\bar{D}^2_w\frac{\delta^3\Gamma}{\delta\bar{c}^{*B}_y\delta V^D_w\delta c^{*C}_z} + \varepsilon\cdot G_c(\partial^2_z/\Lambda^2)D^2_z\frac{\delta^3\Gamma}{\delta\bar{c}^{*B}_y\delta V^C_z\delta c^D_w}$$
$$+\frac{1}{2}G_c\left(\partial^2_y/\Lambda^2\right)D^2_y\frac{\delta^2}{\delta c^{*C}_z\delta c^D_w}\left\langle\delta c^B_y\right\rangle = 0.$$

To simplify these identities we use explicit expressions for the Green functions. They can be derived using dimensional and chirality considerations:

$$\frac{\delta^3\Gamma}{\delta\bar{c}^{*A}_x\delta V^B_y\delta c^C_z} = -\frac{ie_0}{16} f^{ABC}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\Big(f(p,q)\partial^2\Pi_{1/2}$$
$$-F_\mu(p,q)(\gamma^\mu)_{\dot{a}}{}^b\bar{D}^{\dot{a}}D_b + F(p,q)\Big)_y\Big(D^2_x\delta^8_{xy}(q+p)\,\bar{D}^2_z\delta^8_{yz}(q)\Big);$$

$$\frac{\delta^3\Gamma}{\delta\bar{c}^{*A}_x\delta V^B_y\delta c^{*C}_z} = -\frac{ie_0}{16} f^{ABC}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\widetilde{F}(p,q)D^2_x\delta^8_{xy}(q+p)D^2_z\delta^8_{yz}(q),$$

where $\partial^2\Pi_{1/2}\equiv -D^a\bar{D}^2D_a/8$ is the supersymmetric projection operator, and

$$\delta^8_{xy}(p)\equiv\delta^4(\theta_x-\theta_y)e^{ip_\alpha(x^\alpha-y^\alpha)}.$$

This implies that $q+p$ is the momentum of $\bar{c}^*$, $-p$ is the momentum of V, and $-q$ is the momentum of c (or c^*).

Let us introduce the chiral source $\mathcal{J}$ and add the term

$$-\frac{e_0}{2}\int d^4x\, d^2\theta\, f^{ABC}\mathcal{J}^A c^B c^C + \text{c.c.}$$

to the action. From dimensional and chirality considerations we obtain

$$\frac{\delta^2}{\delta c_z^C \delta c_w^D}\left\langle \delta c_y^B \right\rangle = -i\varepsilon \cdot \frac{\delta^3\Gamma}{\delta c_z^C \delta c_w^D \delta \mathcal{J}_y^B}$$

$$= -\frac{ie_0\varepsilon}{4} f^{BCD} \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} H(p,q)\bar{D}_z^2\delta_{zy}^8(q+p)\bar{D}_w^2\delta_{yw}^8(q);$$

where $[H(p,q)] = 1$, and, by construction,

$$H(p,q) = H(p,-q-p).$$

Substituting explicit expressions for the Green functions into the first Slavnov–Taylor identity, we can rewrite it in the form

$$G_c(q)F(q,p) + G_c(p)F(p,q) = 2G_c(q+p)H(-q-p,q);$$

First, let us prove that the function $H(p,q)$ is finite. H is contributed by diagrams in which one leg corresponds to the chiral source $\mathcal{J}$ and two other legs correspond to chiral ghost superfields c. These diagrams contain

$$\int d^4y\, d^2\theta_y\, \mathcal{J}_y^A \cdot \frac{\bar{D}_y^2 D_y^2}{4\partial^2}\delta^8_{y1} \cdot \frac{\bar{D}_y^2 D_y^2}{4\partial^2}\delta^8_{y2} = -2\int d^4y\, d^4\theta_y\, \mathcal{J}_y^A \cdot \frac{D_y^2}{4\partial^2}\delta^8_{y1} \cdot \frac{\bar{D}_y^2 D_y^2}{4\partial^2}\delta^8_{y2}.$$

Therefore, the considered contribution can be presented as an integral over the total superspace, which includes integration over

$$\int d^4\theta = -\frac{1}{2}\int d^2\theta \bar{D}^2 + \text{ total derivatives in the coordinate space }.$$

This implies that two left spinor derivatives should act to the chiral external lines. Therefore, the non-vanishing result can be obtained only if two right spinor derivatives also act to the external lines. Consequently, the result should be proportional to, at least, the second degree of the external momenta and is finite in the ultraviolet region.
Thus, the function $H(p,q)$ is UV finite.

Let us multiply the Slavnov–Taylor identity to the renormalization constant Z_c (such that $(G_c)_R = Z_c G$ is finite), differentiate the result with respect to $\ln\Lambda$, and take the limit $\Lambda\to\infty$. Due to finiteness of $(G_c)_R$ and H the result is written as

$$\Big((G_c)_R(q)\frac{d}{d\ln\Lambda}F(q,p) + (G_c)_R(p)\frac{d}{d\ln\Lambda}F(p,q)\Big)\Big|_{\Lambda\to\infty} = 0.$$

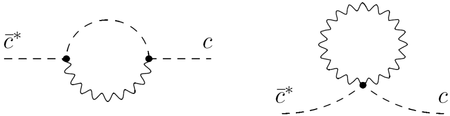
Setting $p=-q$, we obtain

$$\frac{d}{d\ln\Lambda}F(-q,q)\Big|_{\Lambda\to\infty} = 0.$$

Therefore, the corresponding renormalization constant is finite

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0.$$

Thus, the function $F(p,q)$ is also finite. This implies that all three-point ghost-gauge vertices are finite.

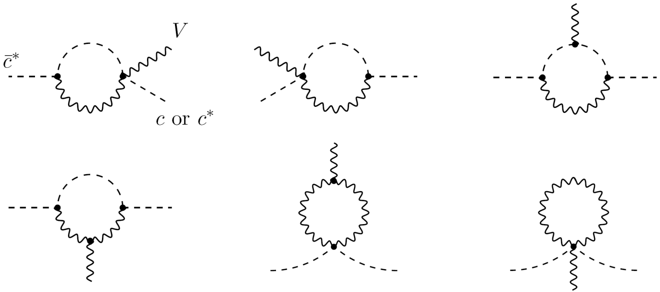


In the Euclidean space after the Wick rotation

$$G_c(p) = 1 + e_0^2 C_2 \int \frac{d^4k}{(2\pi)^4}\Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)\Big(- \frac{1}{6k^4} + \frac{1}{2k^2(k+p)^2} - \frac{p^2}{2k^4(k+p)^2}\Big) + O(e_0^4, e_0^2\lambda_0^2),$$

where $R_k \equiv R(k^2/\Lambda)$ and $K_k \equiv K(k^2/\Lambda^2)$.
We see that this function is divergent in the ultraviolet region (at infinite Λ).

$$\gamma_c(\alpha_0,\lambda_0) = \frac{d\ln G_c}{d\ln\Lambda}\bigg|_{p=0;\ \alpha,\lambda=\text{const}} = -\frac{\alpha_0 C_2(1-\xi_0)}{6\pi} + O(\alpha_0^2, \alpha_0\lambda_0^2).$$



$$\frac{ie_0}{4} f^{ABC} \int d^4\theta\, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \Big(f(p,q) \partial^2 \Pi_{1/2} V^B(\theta,-p)$$
$$+ F_\mu(p,q) (\gamma^\mu)_{\dot{a}}{}^b D_b \bar{D}^{\dot{a}} V^B(\theta,-p) + F(p,q) V^B(\theta,-p) \Big) c^C(\theta,-q);$$
$$\frac{ie_0}{4} f^{ABC} \int d^4\theta\, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \widetilde{F}(p,q) V^B(\theta,-p) c^{*C}(\theta,-q).$$

Calculating these diagrams gives

$$F(p,q) = 1 + \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\Big\{ -\frac{(q+p)^2}{R_k k^2 (k+p)^2 (k-q)^2} - \frac{\xi_0\, p^2}{K_k k^2 (k+q)^2 (k+q+p)^2}$$
$$+ \frac{\xi_0\, q^2}{K_k k^2 (k+p)^2 (k+q+p)^2} + \Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)\Big(-\frac{2(q+p)^2}{k^4 (k+q+p)^2} + \frac{2}{k^2 (k+q+p)^2}$$
$$- \frac{1}{k^2 (k+q)^2} - \frac{1}{k^2 (k+p)^2}\Big)\Big\} + O(\alpha_0^2, \alpha_0\lambda_0^2).$$

$$\widetilde{F}(p,q) = 1 - \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\Big\{ \frac{p^2}{R_k k^2 (k+q)^2 (k+q+p)^2} + \frac{\xi_0\,(q+p)^2}{K_k k^2 (k-p)^2 (k+q)^2}$$
$$+ \frac{\xi_0\, q^2}{K_k k^2 (k+p)^2 (k+q+p)^2} + \frac{2\xi_0}{K_k k^2 (k+p)^2} - \frac{2\xi_0}{K_k k^2 (k+q+p)^2} + \Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)$$
$$\times \Big(\frac{2q^2}{k^4 (k+q)^2} + \frac{1}{k^2 (k+q+p)^2} - \frac{1}{k^2 (k+q)^2}\Big)\Big\} + O(\alpha_0^2, \alpha_0\lambda_0^2).$$

We see that these expressions are finite in the ultraviolet region.

The expressions for the functions f and F_μ are very large and in writing them we will use the notation

$$\Delta_q \equiv \frac{\xi_0}{K_q} - \frac{1}{R_q}.$$

The function f has the form

$$f(p,q) = \frac{1}{4}\int \frac{d^4k}{(2\pi)^4}\,\frac{e_0^2 C_2}{k^2(k+q)^2(k+q+p)^2}\Big\{\frac{2k_\mu q_\mu}{(k+q)^2}\Delta_{k+q} + \frac{2k^2}{(k+q+p)^2}\Delta_{k+q+p}$$
$$+R_p\Big(\frac{2k_\mu(q+p)^\mu}{(k+q+p)^2 R_{k+q}}\Delta_{k+q+p} + \frac{2k^2}{(k+q)^2 R_{k+q+p}}\Delta_{k+q} + \Big(\frac{k_\mu(k+q+p)^\mu}{(k+q+p)^2}$$
$$+\frac{k_\mu(k+q)^\mu}{(k+q)^2}\Big)\Delta_{k+q}\Delta_{k+q+p}\Big) - \frac{2k_\mu(k+q)^\mu}{R_{k+q}R_{k+q+p}}\cdot\frac{R_{k+q+p}-R_{k+q}}{(k+q+p)^2-(k+q)^2}$$
$$-\frac{2(R_{k+q+p}-R_p)}{(k+q+p)^2-p^2}\cdot\frac{1}{R_{k+q+p}}\Big(\frac{k_\mu q^\mu(k+q+p)^2 - k_\mu q^\mu p^2}{(k+q)^2}\Delta_{k+q} + \frac{k_\mu p^\mu}{R_{k+q}}\Big)$$
$$-\frac{2(R_{k+q}-R_p)}{(k+q)^2-p^2}\cdot\frac{1}{R_{k+q}}\Big(\frac{k^2(k+q)^2 - k^2p^2}{(k+q+p)^2}\Delta_{k+q+p} + \frac{k_\mu(k+q)^\mu}{R_{k+q+p}}\Big)\Big\} + O(e_0^4, e_0^2\lambda_0^2).$$

$$F_\mu(p,q) = \frac{1}{16}\int \frac{d^4k}{(2\pi)^4}\frac{e_0^2 C_2}{k^2(k+q)^2(k+q+p)^2}\Big\{\frac{2}{k^2}\Delta_k\Big[(q+p)_\mu\, k_\alpha (k+q)^\alpha + q_\mu\, k_\alpha$$

$$\times (k+q+p)^\alpha + k_\mu\Big(k^2 - q^2 - q_\alpha p^\alpha\Big)\Big] - \frac{4k_\mu}{R_{k+q}} + \frac{2}{(k+q)^2}\Delta_{k+q}\Big[- q_\mu k_\alpha p^\alpha + p_\mu k^2$$

$$+ k_\mu q_\alpha p^\alpha - k_\mu (k+q)^2 + k_\alpha q^\alpha (2q+2k+p)_\mu\Big] + \frac{2}{(k+q+p)^2}\Delta_{k+q+p}\Big[q_\mu k_\alpha (q+p)^\alpha$$

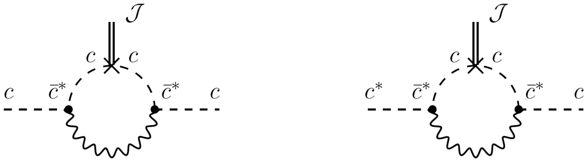
$$+ (q+p)_\mu k_\alpha q^\alpha - k_\mu (q^2 + q_\alpha p^\alpha + k^2) - p_\mu k^2\Big] - \frac{R_{k+q+p} - R_{k+q}}{(k+q+p)^2 - (k+q)^2}(2q+2k+p)_\mu$$

$$\times \frac{4k^\alpha q_\alpha}{R_{k+q}R_{k+q+p}} + \frac{2R_p}{(k+q)^2(k+q+p)^2}\Delta_{k+q+p}\Delta_{k+q}\Big[(p_\mu p^\nu - \delta_\mu^\nu p^2)\Big((k^2+q^2)(k_\nu + q_\nu)$$

$$- (k+q)^2 q_\nu\Big) + p^2 (q_\mu k_\alpha p^\alpha - k_\mu q_\alpha p^\alpha)\Big] + \frac{4R_p}{(k+q)^2 R_{k+q+p}}\Delta_{k+q}\,(q_\mu k_\alpha p^\alpha - k_\mu q_\alpha p^\alpha)$$

$$+ \frac{4(R_{k+q} - R_p)}{(k+q)^2 - p^2}\,\frac{(k_\mu q_\alpha p^\alpha - q_\mu k_\alpha p^\alpha)}{R_{k+q}R_{k+q+p}} + \frac{4(R_{k+q+p} - R_p)}{(k+q+p)^2 - p^2}\Big(\frac{(p_\mu p^\nu - \delta_\mu^\nu p^2)k_\nu}{R_{k+q+p}R_{k+q}} + \Delta_{k+q}$$

$$\times \frac{\big((k+q+p)^2 - p^2\big)}{(k+q)^2 R_{k+q+p}}\Big(q_\mu k_\alpha p^\alpha - k_\mu q_\alpha p^\alpha\Big)\Big)\Big\} + O(e_0^4, e_0^2\lambda_0^2).$$



$$H(p,q) = 1 - \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\Bigg\{\frac{p^2}{R_k k^2 (k+q)^2 (k+q+p)^2} + \frac{(q+p)^2}{k^4 (k+q+p)^2}\Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big) + \frac{q^2}{k^4(k+q)^2}\Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)\Bigg\} + O(e_0^4, e_0^2\lambda_0^2);$$

$$\widetilde H(p,q) = \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\frac{1}{K_k k^2 (k+q)^2 (k+q+p)^2} + O(e_0^4, e_0^2\lambda_0^2).$$

We see that the function H is finite in the ultraviolet region and is quadratic in external momenta.

We will define the RG functions in terms of the bare couplings by the equations

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda};$$

$$(\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d\ln(Z_\phi)_i{}^j(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda};$$

$$\gamma_V(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_V(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda};$$

$$\gamma_c(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_c(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}.$$

where the differentiation is made at fixed values of α and λ^{ijk}.

There renormalization group functions are

1. scheme independent at a fixed regularization;
2. depend on a regularization;
2. satisfy the NSVZ relation in all orders for $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives.

The NSVZ β-function can be equivalently rewritten in the form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

Let us express the β-function in the right hand side in terms of the renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}.$$

Then, using the identity $d(Z_\alpha^{-1/2}Z_V Z_c)/d\ln\Lambda = 0$ we obtain

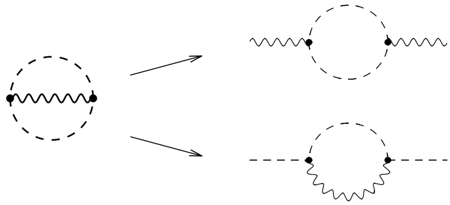
$$\beta(\alpha_0,\lambda_0) = -2\alpha_0\frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0)+\gamma_V(\alpha_0,\lambda_0)\Big),$$

where γ_c and γ_V are anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare coupling constants), respectively.

Substituting this expression into the right hand side of the NSVZ relation we obtain

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

From this form of the NSVZ β-function we see that the matter superfields and ghosts similarly contribute to the right hand side.



Renormalization group functions defined in terms of the renormalized couplings

The RG functions are defined in terms of the renormalized couplings by the equations

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu};$$

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) \equiv \frac{d\ln(Z_\phi)_i{}^j(\alpha_0,\lambda_0,\Lambda/\mu)}{d\ln\mu};$$

$$\widetilde{\gamma}_V(\alpha,\lambda) \equiv \frac{d\ln Z_V(\alpha_0,\lambda_0,\Lambda/\mu)}{d\ln\mu};$$

$$\widetilde{\gamma}_c(\alpha,\lambda) \equiv \frac{d\ln Z_c(\alpha_0,\lambda_0,\Lambda/\mu)}{d\ln\mu}.$$

where the differentiation is made at fixed values of α_0 and λ_0^{ijk}.
There renormalization group functions are
1. scheme and regularization dependent;
2. satisfy the NSVZ relation only for a special renormalization prescription, called the NSVZ scheme.

The RG functions defined in terms of the renormalized coupling constant are scheme dependent and satisfy the NSVZ relation only in a certain subtraction scheme. Similarly to

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459; Phys.Lett. **B730** (2014) 184.

we see that in the non-Abelian case the RG functions defined in terms of the bare coupling constant coincide with ones defined in terms of the renormalized coupling constants if the boundary conditions

$$Z_\alpha(\alpha,\lambda,x_0)=1;\qquad (Z_\phi)_i{}^j(\alpha,\lambda,x_0)=\delta_i{}^j;\qquad Z_c(\alpha,\lambda,x_0)=1,$$

where x_0 is a fixed value of $\ln\Lambda/\mu$, are imposed on the renormalization constants. (For $x_0=0$ we obtain minimal subtractions.) We also assume that the renormalization constants satisfy the equation

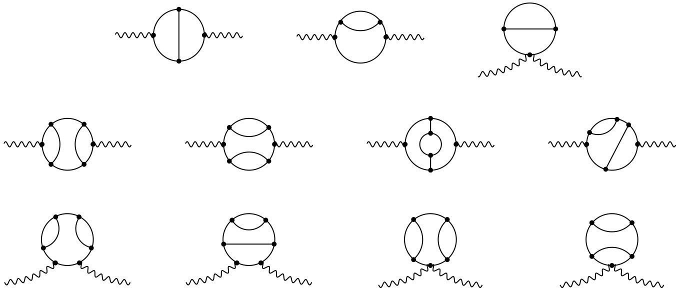
$$Z_V=Z_\alpha^{1/2}Z_c^{-1},$$

Possibly, these conditions give the NSVZ scheme with the higher covariant derivative regularization.

To verify the above results we consider the three-loop terms quartic in the Yukawa couplings. They correspond to the graphs

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345.

Attaching two external lines of the background gauge superfield we obtain the diagrams contributing to the β-function.



The corresponding contribution to the anomalous dimension is given by the diagrams

The calculation gives the following result:

$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r} C(R)_i{}^j \frac{d}{d\ln\Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}\, \lambda_0^{imn}\lambda^*_{0jmn} \frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}$$

$$\times\Big(\frac{1}{k^2F_k\, q^2F_q\,(q+k)^2F_{q+k}}\Big) + \frac{4\pi}{r} C(R)_i{}^j \frac{d}{d\ln\Lambda}\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}$$

$$\times\Bigg(\lambda_0^{iab}\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k^\mu} - \frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}\Big) + 2\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q^\mu}\Bigg)$$

$$\times\frac{1}{k^2F_k^2\, q^2F_q\,(q+k)^2F_{q+k}\, l^2F_l\,(l+k)^2F_{l+k}} = -\frac{1}{2\pi r} C(R)_i{}^j \Delta\gamma_\phi(\lambda_0)_j{}^i.$$

The simplest regulator has the form $F(k^2/\Lambda^2) = 1 + k^2/\Lambda^2$. In this case

$$\Delta\gamma_\phi(\alpha_0,\lambda_0)_j{}^i = \frac{1}{4\pi^2}\lambda_0^{iab}\lambda^*_{0jab} - \frac{1}{16\pi^4}\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde};$$

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) - \frac{1}{2\pi r}C(R)_i{}^j\Delta\gamma_\phi(\lambda_0)_j{}^i + O(\alpha_0) + O(\lambda_0^6).$$

and the NSVZ relation is valid for the RG functions defined in terms of the bare couplings with the HD regularization.

$$(\ln Z_\phi)_j{}^i = -\frac{1}{4\pi^2}\lambda_0^{iab}\lambda^*_{0jab}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{1}{32\pi^4}\lambda_0^{iab}\lambda^*_{0kab}\lambda_0^{kcd}\lambda^*_{0jcd}\Big(\ln^2\frac{\Lambda}{\mu}$$
$$+2g_1\ln\frac{\Lambda}{\mu} + 2g_1^2 - \widetilde{g}_2\Big) + \frac{1}{16\pi^4}\lambda_0^{iab}\lambda^*_{0jac}\lambda_0^{cde}\lambda^*_{0bde}\Big(\ln\frac{\Lambda}{\mu} + \ln^2\frac{\Lambda}{\mu} + 2g_1\ln\frac{\Lambda}{\mu}$$
$$+2g_1^2 - g_2\Big) + O(\alpha_0) + O(\lambda_0^6);$$

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} + \frac{1}{2\pi}\Big(3C_2 - T(R)\Big)\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) + \frac{1}{2\pi r}C(R)_i{}^j\Big[\frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big)$$
$$+\frac{1}{32\pi^4}\lambda^{iab}\lambda^*_{kab}\lambda^{kcd}\lambda^*_{jcd}\Big(\ln^2\frac{\Lambda}{\mu} + 2g_1\ln\frac{\Lambda}{\mu} + \widetilde{b}_3\Big) + \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde}\Big(-\ln\frac{\Lambda}{\mu}$$
$$+\ln^2\frac{\Lambda}{\mu} + 2g_1\ln\frac{\Lambda}{\mu} + b_3\Big)\Big] + O(\alpha) + O(\lambda^6).$$

The RG functions defined in terms of the renormalized couplings are

$$\widetilde{\gamma}_\phi(\alpha,\lambda)_j{}^i = \frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab} - \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde} + O(\alpha) + O(\lambda^6).$$

$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{1}{2\pi r}C(R)_i{}^j\Big[-\frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab} + \frac{1}{16\pi^4}$$
$$\times\lambda^{iab}\lambda^*_{kab}\lambda^{kcd}\lambda^*_{jcd}\Big(b_2 - g_1\Big) + \frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde}\Big(1 + 2b_2 - 2g_1\Big)\Big]$$
$$+O(\alpha) + O(\lambda^6).$$

We see that the considered part of this β-function is scheme-dependent. Imposing the boundary conditions

$$Z_\phi(\alpha,\lambda,x_0)_i{}^j = \delta_i{}^j; \qquad Z_\alpha(\alpha,\lambda,x_0) = \alpha/\alpha_0 = 1$$

we obtain $g_1 = b_1 = b_2 = -x_0$. Therefore, $b_2 - g_1 = 0.$

This implies that the NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

is really valid for the RG functions defined in terms of the renormalized couplings,

$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{1}{2\pi r}C(R)_i{}^j\Big[-\frac{1}{4\pi^2}\lambda^{iab}\lambda^*_{jab}$$
$$+\frac{1}{16\pi^4}\lambda^{iab}\lambda^*_{jac}\lambda^{cde}\lambda^*_{bde}\Big] + O(\alpha) + O(\lambda^6)$$
$$= -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) - \frac{1}{2\pi r}C(R)_i{}^j\widetilde{\gamma}_\phi(\alpha,\lambda)_i{}^j + O(\alpha) + O(\lambda^6).$$

Also we see that the NSVZ scheme is actually obtained with the higher covariant derivative regularization supplemented by minimal subtractions.

Conclusion

- ✓ In the case of using the higher covariant derivative regularization the integrals defining the β-function are integrals of double total derivatives in the momentum space. This has been proved in some theories in all loops. For general non-Abelian SYM there are strong evidences in favour of this.
- ✓ The factorization into double total derivatives naturally leads to the NSVZ relation for RG functions defined in terms of the bare coupling constant, which is valid independently of the subtraction scheme with the HD regularization.
- ✓ For RG functions defined in terms of the renormalized coupling constant the NSVZ scheme can be constructed by imposing simple boundary conditions on the renormalization constants in the case of using the HD regularization. The NSVZ scheme obtained in this way can be considered as minimal subtractions.
- ✓ The non-trivial three-loop calculation for the terms quartic in the Yukawa couplings confirms this proposal for the NSVZ scheme.
- ✓ For $\mathcal{N}=1$ SYM the derivation of the NSVZ relation seems to involve the non-renormalization theorem for the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield.

Thank you for the attention!